EXPLICIT CONSTRUCTION OF HARMONIC TWO-SPHERES INTO THE COMPLEX GRASSMANNIAN

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ABSTRACT. We present an explicit description of all harmonic maps of finite uniton number from a Riemann surface into a complex Grassmannian. Namely, starting from a constant map Q and a collection of meromorphic functions and their derivatives, we show how to algebraically construct all harmonic maps from the two-sphere into a given Grassmannian $G_p(\mathbb{C}^n)$. In this setting the uniton number depends on Q and p and we obtain a sharp estimate for it.

INTRODUCTION

Harmonic spheres in complex Grassmannians have been extensively studied using various techniques (see [2, 4, 5]). As it is well-known, the complex Grassmannian sits naturally in the unitary group $\mathbf{U}(N)$ equipped with its standard bi-invariant metric, via its Cartan totally geodesic embedding. Using a Bäcklund transformation technique, Uhlenbeck [14] obtained a method to construct successive harmonic maps into $\mathbf{U}(N)$ from an initial harmonic map. She proved that through this process, called "adding a uniton", one can obtain all harmonic maps from a Riemann surface with finite uniton number. Subsequent works have expanded this view. However obtaining explicit unitons involves solving $\overline{\partial}$-problems which is a difficult task [10, 15]. In [3] J. C. Wood and the authors gave an algebraic procedure to construct these unitons so that one can build all harmonic maps with finite uniton number from a Riemann surface into $\mathbf{U}(N)$, from freely chosen meromorphic functions into $\mathbb{C}^n$ and their derivatives. Although these harmonic maps include those with values in the Grassmannian, no explicit way was given to decide when, from a specific meromorphic data, one could obtain a Grassmannian-valued harmonic map. The aim of this paper is to study, from this point of view, harmonic maps with finite uniton number from a Riemann surface into a Grassmannian manifold. More specifically, we present algebraic conditions, to be satisfied by the initial data, ensuring that the obtained harmonic maps have values in a Grassmannian (Theorem 2.5). Furthermore, for a specific Grassmannian manifold $G_p(\mathbb{C}^n)$, we show how to organize our initial data so that the harmonic map has its image in the given Grassmannian manifold (Theorem 2.17).

Associated to a harmonic map $\phi : M^2 \to \mathbf{U}(n)$, there is a spectral deformation, called the extended solution; that is a family of maps $\Phi_\lambda : M^2 \to \mathbf{U}(n)$, depending smoothly on $\lambda \in S^1$, such that $\phi = Q\Phi_{-1}$ (for some $Q \in \mathbf{U}(n)$) and the differential form $A^\lambda = \frac{1}{2}\Phi_\lambda^{-1} d\Phi_\lambda$ satisfies [14]

$$A^\lambda = \frac{1}{2}(1-\lambda^{-1})A_z^\lambda + \frac{1}{2}(1-\lambda)A_{\bar{z}}^\lambda.$$

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The extended solution is not, in general, unique. However, Uhlenbeck proved that, given a harmonic map, there exists a unique extended solution Φ_λ of type-one, i.e., such that the image of Φ_0 is full. Furthermore, given a harmonic map $\phi : M^2 \to G_p(\mathbb{C}^n)$ with finite uniton number, there exists $Q = \pi_{F_0} - \pi_{F_0}^\perp \in \mathbf{U}(n)$, such that $\phi = Q\Phi_{-1}$, where Φ_λ denotes the type-one extended solution, and π_{F_0} denotes the orthogonal projection onto a complex subspace F_0 of $\mathbb{C}^n$. Under these conditions, we present an estimate for the uniton number of such a harmonic map, depending on p and Q. This estimate is sharp. It is known that, for a harmonic map $\phi : M^2 \to G_p(\mathbb{C}^n)$, the maximal uniton number is less or equal than $2\min\{p, n-p\}$ ([1, 7]). We show that this value is only attained when $Q = \pm I$. Unlike the case of harmonic maps $\phi : S^2 \to \mathbf{U}(n)$, for $G_p(\mathbb{C}^n)$-valued harmonic maps, the possible uniton numbers depend on Q. In [13], G. Segal gave a model for the loop group of $\mathbf{U}(n)$ as an (infinite-dimensional) Grassmannian and showed that harmonic maps of finite uniton number correspond to holomorphic maps into a related finite-dimensional Grassmannian. We interpret our results in the framework of the Grassmannian model and relate them with those in [3].
The paper is organized as follows: in Section 1 we recall Uhlenbeck's factorization an explain the algebraic procedure, presented in [3], to construct explicit unitons. Section 2 is devoted to the study of Grassmannian-valued harmonic maps. In 2.1 we describe the main results for harmonic maps $\phi : S^2 \to G_*(\mathbb{C}^n)$ and present examples. Harmonic maps with values in a specific Grassmannian manifold are treated in 2.2. Subsection 2.3 is devoted to the interpretation of our construction in the Grassmannian model setting. All involved calculations and proofs are presented, separately, in Subsection 2.4.

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1. Preliminaries: harmonic maps into $\mathbf{U}(n)$

Let M^2 be a Riemann surface. For any smooth map $\phi : M^2 \to \mathbf{U}(n)$, let A^ϕ denote one half the pull-back of the Maurer–Cartan form,

$$A^\phi = \tfrac{1}{2}\phi^{-1}\mathrm{d}\phi. \tag{1.1}$$

Choosing a local complex coordinate z on an open subset of M^2, we write $A^\phi = A^\phi_z \mathrm{d}z + A^\phi_{\bar z}\mathrm{d}\bar z$, where A^ϕ_z and $A^\phi_{\bar z}$ denote the $(1,0)$- and $(0,1)$-parts (with respect to M^2), respectively. Let $\underline{\mathbb{C}}^n$ denote the trivial complex bundle $M^2 \times \mathbb{C}^n$ equipped with the standard Hermitian inner product: $< u, v >= u_1\overline{v}_1 + \cdots + u_n\overline{v}_n$ $(u = (u_1, \ldots, u_n),\ v = (v_1, \ldots, v_n) \in \mathbb{C}^n)$ on each fibre. A^ϕ_z and $A^\phi_{\bar z}$ are local sections of the endomorphism bundle $\mathrm{End}(\underline{\mathbb{C}}^n)$, and each is minus the adjoint of the other. $D^\phi := \mathrm{d} + A^\phi$ is a unitary connection on the trivial bundle $\underline{\mathbb{C}}^n$; in fact, it is the pull-back of the Levi-Civita connection $\mathbf{U}(n)$.
We write $D^\phi_z = \partial_z + A^\phi_z$ and $D^\phi_{\bar z} = \partial_{\bar z} + A^\phi_{\bar z}$ where $\partial_z = \partial/\partial z$ and $\partial_{\bar z} = \partial/\partial \bar z$ for a (local) complex coordinate z on M^2. Give $\underline{\mathbb{C}}^n$ the Koszul–Malgrange complex structure [11]; this is the unique holomorphic structure such that a (local) section σ of $\underline{\mathbb{C}}^n$ is holomorphic if and only if $D^\phi_{\bar z}\sigma = 0$ for any complex coordinate z; we shall denote the resulting holomorphic bundle by $(\underline{\mathbb{C}}^n, D^\phi_{\bar z})$. Note that, when ϕ is constant, $A^\phi = 0$, and the Koszul-Malgrange holomorphic structure is the standard holomorphic structure on $\underline{\mathbb{C}}^n$.

Since A_z^ϕ represents the derivative $\partial\phi/\partial z$, *the map* ϕ *is harmonic if and only if the endomorphism* A_z^ϕ *is holomorphic*, i.e.,

$$A_z^\phi \circ D_{\bar z}^\phi = D_{\bar z}^\phi \circ A_z^\phi \,.$$

Let $\phi : M^2 \to \mathbf{U}(n)$ be harmonic and let $\underline{\alpha}$ be a smooth subbundle of $\underline{\mathbb{C}}^n$. We shall say that $\underline{\alpha}$ is *proper* if it is neither the zero subbundle nor the full bundle $\underline{\mathbb{C}}^n$ and we consider that $\underline{\alpha}$ is *full* if is not contained in any proper subspace of $\mathbb{C}^n$. Finally, by a *uniton* or *flag factor for* ϕ we mean a smooth subbundle $\underline{\alpha}$ such that

$$\left\{ \begin{array}{l} \text{(i) } D_{\bar z}^\phi(\sigma) \in \Gamma(\underline{\alpha}) \text{ for all } \sigma \in \Gamma(\underline{\alpha})\,, \\ \text{(ii) } A_z^\phi(\sigma) \in \Gamma(\underline{\alpha}) \text{ for all } \sigma \in \Gamma(\underline{\alpha})\,; \end{array} \right. \tag{1.2}$$

here $\Gamma(\cdot)$ denotes the space of smooth sections of a bundle. These equations say that $\underline{\alpha}$ is a holomorphic subbundle of $(\underline{\mathbb{C}}^n, D_{\bar z}^\phi)$ which is closed under the endomorphism A_z^ϕ.
For a subbundle $\underline{\alpha}$ of $\underline{\mathbb{C}}^n$, let π_α and $\pi_\alpha^\perp$ denote orthogonal projection onto $\underline{\alpha}$ and onto its orthogonal complement $\underline{\alpha}^\perp$, respectively. Then [14]

Proposition 1.1. *The map* $\widetilde{\phi} : M^2 \to \mathbf{U}(n)$ *given by* $\widetilde{\phi} = \phi(\pi_\alpha - \pi_\alpha^\perp)$ *is harmonic if and only if* $\underline{\alpha}$ *is a uniton.*

Note that $\underline{\alpha}$ is a uniton for ϕ if and only if $\underline{\alpha}^\perp$ is a uniton for $\widetilde{\phi}$; further $\phi = -\widetilde{\phi}(\pi_\alpha^\perp - \pi_\alpha)$ i.e., the flag transforms defined by $\underline{\alpha}$ and $\underline{\alpha}^\perp$ are inverse up to sign.
Given a harmonic map ϕ and a uniton $\underline{\alpha}$ for ϕ, we can characterize the holomorphic structure $D_{\bar z}^{\widetilde{\phi}}$ as well as the operator $A_z^{\widetilde{\phi}}$ for the new harmonic map $\widetilde{\phi} = \phi(\pi_\alpha - \pi_\alpha^\perp)$ by the simple formulae [14]

$$A_z^{\widetilde{\phi}} = A_z^\phi + \partial_z \pi_\alpha^\perp \,, \quad D_{\bar z}^{\widetilde{\phi}} = D_{\bar z}^\phi - \partial_{\bar z} \pi_\alpha^\perp .$$

Hence, we can also write down the uniton equations (1.2) for the harmonic map $\widetilde{\phi}$. In general, finding unitons for the harmonic map $\widetilde{\phi}$ would require to solve a $\overline{\partial}$-problem. However, the following result ([3], Theorem 1.1.) gives an explicit construction of these unitons (for a different approach, see [6]).

Theorem 1.2. *For any* $r \in \{0, 1, \ldots, n-1\}$, *let* $(H_{i,j})_{0 \le i \le r-1, 1 \le j \le n}$ *be an* $r \times n$ *array of* $\mathbb{C}^n$*-valued meromorphic functions on* M^2, *and let* ϕ_0 *be an element of* $\mathbf{U}(n)$. *For each* $i = 0, 1, \ldots, r-1$, *set* $\underline{\alpha}_{i+1}$ *equal to the subbundle of* $\underline{\mathbb{C}}^n$ *spanned by the vectors*

$$\alpha_{i+1,j}^{(k)} = \sum_{s=k}^{i} C_s^i H_{s-k,j}^{(k)} \qquad (j = 1, \ldots, n,\ k = 0, 1, \ldots, i). \tag{1.3}$$

Then, the map $\phi : M^2 \to \mathbf{U}(n)$ *defined by*

$$\phi = \phi_0(\pi_1 - \pi_1^\perp) \cdots (\pi_r - \pi_r^\perp) \tag{1.4}$$

is harmonic.
Further, all harmonic maps of finite uniton number, and so all harmonic maps from S^2, *are obtained this way.*

In the above result, by a $\mathbb{C}^n$-*valued meromorphic function* or *meromorphic vector* H on M^2, it is simply meant an n-tuple of meromorphic functions; its k'th derivative with respect to some local complex coordinate on M^2 is denoted by $H^{(k)}$. Also, π_i denotes $\pi_{\underline{\alpha}_i}$ whereas $\pi_i^\perp$ stands for $\pi_{\underline{\alpha}_i^\perp}$. Moreover, for integers i and s with $0 \leq s \leq i$, C_s^i denotes the *s'th elementary function* of the projections $\pi_i^\perp, \ldots, \pi_1^\perp$ given by

$$C_s^i = \sum_{1 \leq i_1 < \cdots < i_s \leq i} \pi_{i_s}^\perp \cdots \pi_{i_1}^\perp. \tag{1.5}$$

C_s^i denotes the identity when $s = 0$ and zero when $s < 0$ or $s > i$. Note that the C_s^i satisfy a property like that for Pascal's triangle

$$C_s^i = \pi_i^\perp C_{s-1}^{i-1} + C_s^{i-1} \quad (i \geq 1, 0 \leq s \leq i). \tag{1.6}$$

Moreover, the unitons $\underline{\alpha}_i$ satisfy the *covering condition*

$$\pi_i \underline{\alpha}_{i+1} = \underline{\alpha}_i \qquad (i = 1, \ldots, r-1). \tag{1.7}$$

We quickly review the main steps in the proof of the above theorem (for more details, we refer the reader to [3]). To see that the map ϕ in (1.4) is harmonic, all one has to do is to check that the successive bundles $\underline{\alpha}_{i+1}$ satisfy equations (1.2) for each of the harmonic maps ϕ_i, where

$$\phi_i = \phi_0(\pi_1 - \pi_1^\perp) \cdots (\pi_i - \pi_i^\perp).$$

This follows from an explicit calculation showing that ([3], Proposition 2.4.):
(i) $\alpha_{i+1,j}^{(k)}$ are holomorphic sections of $(\underline{\mathbb{C}}^n, D_{\bar{z}}^{\phi_i})$ and
(ii) $A_z^{\phi_i}(\alpha_{i+1,j}^{(k)}) = \begin{cases} -\alpha_{i+1,j}^{(k+1)}\,, & \text{if } k < i+1\,, \\ 0\,, & \text{if } k = i+1\,; \end{cases}$
As for the converse, one needs to develop further the theory. Let A^ϕ be as in (1.1) and set

$$A^\lambda = \tfrac{1}{2}(1 - \lambda^{-1}) A_z^\phi \mathrm{d}z + \tfrac{1}{2}(1 - \lambda) A_{\bar{z}}^\phi \mathrm{d}\bar{z} \qquad (\lambda \in S^1).$$

It is well-known that the harmonicity of ϕ implies the integrability of A^λ and we can therefore find, at least locally, an S^1-family of smooth maps $\Phi = \Phi_\lambda : M^2 \to \mathbf{U}(n)$ with

$$\tfrac{1}{2}\Phi_\lambda^{-1}\mathrm{d}\Phi_\lambda = A^\lambda \qquad (\lambda \in S^1) \qquad \text{and} \qquad \Phi_1(z) = I \text{ for all } z \in M^2,$$

where I is the identity matrix. We say that $\Phi = \Phi_\lambda : M^2 \to \mathbf{U}(n)$ is an *extended solution* [14] (for ϕ) and it is clear that Φ can be interpreted as a map into a loop group.
Note that any two extended solutions for a harmonic map differ by a function ('constant loop') $Q : S^1 \to \mathbf{U}(n)$ with $Q(1) = 1$. Further, Φ_{-1} is left-equivalent to ϕ, i.e., $\Phi_{-1} = Q\phi$ for some constant $Q \in \mathbf{U}(n)$.
Let $\mathfrak{gl}(n, \mathbb{C})$ denote the Lie algebra of $n \times n$ matrices; this is the complexification of $\mathfrak{u}(n)$. The extended solution extends to a family of maps $\Phi_\lambda : M^2 \to \mathfrak{gl}(n, \mathbb{C})$ with Φ_λ a holomorphic function of $\lambda \in \mathbb{C} \setminus \{0\}$. Hence it can be expanded as a Laurent series, $\Phi = \sum_{i=-\infty}^{\infty} \lambda^i T_i$, where each $T_i = T_i^\Phi$ is a smooth map from M^2 to $\mathfrak{gl}(n, \mathbb{C})$.
A harmonic map $\phi : M^2 \to \mathbf{U}(n)$ is said to be *of finite uniton number* if it has a *polynomial* extended solution

$$\Phi = T_0 + \lambda T_1 + \cdots + \lambda^r T_r. \tag{1.8}$$

The *(minimal) uniton number* of ϕ is the least degree of all its polynomial extended solutions. In general, given a harmonic map ϕ (with finite uniton number r) there is not a unique corresponding extended solution Φ with degree r. Nevertheless, if one further imposes that the subbundle $\underline{\mathrm{Im}}T_0$ is full, uniqueness is achieved [14]. Such extended solutions have a unique factorization

$$\Phi = (\pi_1 + \lambda\pi_1^\perp)\cdots(\pi_r + \lambda\pi_r^\perp) \tag{1.9}$$

where $\underline{\alpha}_1, ..., \underline{\alpha}_r$ are proper unitons satisfying the covering condition (1.7) and $\underline{\alpha}_1$ is full; these will be called *type-one* extended solutions. One can then prove that each of these subbundles $\underline{\alpha}_i$ is of the form stated in Theorem 1.2.

Example 1.3. [3] Let $\phi : M^2 \to \mathbf{U}(3)$ be a non-constant harmonic map of finite uniton number. Then, *either*

(a) it has uniton number one and is given by a holomorphic map $\phi : M^2 \to G_{d_1}(\mathbb{C}^3)$ where $d_1 = 1$ or 2; *or*

(b) it has uniton number two and is given by (1.3) with unitons $\underline{\alpha}_1$, $\underline{\alpha}_2$ of rank one and two respectively and $\underline{\alpha}_1$ full. The data of Theorem 1.2 consists of maps $H_{0,1}$ and $H_{1,1}$. Then, since $A_z^{\phi_1}(H_{0,1}) = -\pi_1^\perp H_{0,1}^{(1)}$,

$$\underline{\alpha}_1 = \mathrm{span}\{H_{0,1}\} \quad \text{and} \quad \underline{\alpha}_2 = \mathrm{span}\{H_{0,1} + \pi_1^\perp H_{1,1}, \pi_1^\perp H_{0,1}^{(1)}\}. \tag{1.10}$$

2. Harmonic maps into $G_*(\mathbb{C}^n)$

2.1. Explicit construction.

For any $p \in \{0, 1, \ldots, n\}$, let $G_p(\mathbb{C}^n)$ denote the *complex Grassmannian* of k-dimensional subspaces of $\mathbb{C}^n$ equipped with its standard structure as a Hermitian symmetric space. It is convenient to denote the disjoint union $\cup_{p=0}^n G_p(\mathbb{C}^n)$ by $G_*(\mathbb{C}^n)$. In the sequel, we always identify a map into $G_p(\mathbb{C}^n)$ with the pull-back of the corresponding tautological bundle. As it is well-known, $G_p(\mathbb{C}^n)$ sits totally geodesically in $\mathbf{U}(n)$ via the *Cartan embedding* $\iota(F) = \pi_F - \pi_F^\perp$. The formulae in Theorem 1.2 gives all harmonic maps from S^2 into $G_*(\mathbb{C}^n)$, although it does not tell us *how to choose the holomorphic data* $H_{i,j}$ in order to guarantee that the resulting map ϕ lies in $G_*(\mathbb{C}^n)$. On the other hand the situation is now somehow different, in the sense that left multiplication by a constant map Q does not, in general, preserve the image in $G_*(\mathbb{C}^n)$ [14]. Therefore the classification up to left multiplication is no longer suitable in this setting.

Example 2.1. Let $\phi : M^2 \to G_*(\mathbb{C}^n)$ be a non-constant harmonic map of uniton number one. Then, if ϕ is not holomorphic, it must be of the form
$\phi = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)$.
It is easily seen that ϕ is $G_*(\mathbb{C}^n)$-valued if, and only if, π_{F_0} and π_1 commute; equivalently, F_0 *decomposes* $\underline{\alpha}_1$; i.e.,

$$\underline{\alpha}_1 = \underline{\alpha}_1 \cap F_0 \oplus \underline{\alpha}_1 \cap F_0^\perp.$$

In that case, we can easily check that

$$\phi = \pi_{F_1} - \pi_{F_1}^\perp$$

where

$$\underline{F}_1 = \underline{\alpha}_1 \cap F_0 \oplus \underline{\alpha}_1^\perp \cap F_0^\perp. \tag{2.1}$$

Notice that if F_0 is not trivial and $\underline{\alpha}_1$ is full, then it must be that rank $\underline{\alpha}_1 \geq 2$. Moreover, in that case, ϕ_1 decomposes into $\phi_1 \cap \underline{\alpha}_1$ and $\phi_1 \cap \underline{\alpha}_1^\perp$ which are, respectively, holomorphic and anti-holomorphic subbundles of $(\underline{\mathbb{C}}^n, \partial_{\bar{z}})$.

Example 2.2. A harmonic map $\phi : M^2 \to G_*(\mathbb{C}^n)$ with uniton number 2 can be written as $\phi = (\pi_{F_0} - \pi_{F_0^\perp})(\pi_1 - \pi_1^\perp)(\pi_2 - \pi_2^\perp)$, where F_0 is a complex subspace of $\mathbb{C}^n$ and $\underline{\alpha}_1$ is full. From ([14], Theorem 15.3) we know that $\phi_1 = (\pi_{F_0} - \pi_{F_0^\perp})(\pi_1 - \pi_1^\perp)$ must be also Grassmannian-valued. As in Example 2.1, F_0 splits $\underline{\alpha}_1$ and $\phi_1 = \pi_{F_1} - \pi_{F_1^\perp}$, where $\underline{F}_1$ is given by (2.1). Again, since ϕ has values in $G_*(\mathbb{C}^n)$, π_2 and π_{F_1} commute which implies that $\underline{F}_1$ splits $\underline{\alpha}_2$ and $\phi = \pi_{F_2} - \pi_{F_2}^\perp$, where

$$\underline{F}_2 = \underline{\alpha}_2 \cap \underline{F}_1 \oplus \underline{\alpha}_2^\perp \cap \underline{F}_1^\perp. \tag{2.2}$$

When F_0 is trivial (i.e. $F_0 = \mathbb{C}^n$ or $F_0 = \{0\}$), it is easily seen, from the covering condition and the fact that π_1 and π_2 commute, that $\underline{\alpha}_1 \subset \underline{\alpha}_2$. Hence, according to Theorem 1.2,

$$\begin{aligned}
\underline{\alpha}_1 &= \operatorname{span}\{H_{0,1}, ..., H_{0,r}\}\\
\underline{\alpha}_2 &= \operatorname{span}\{H_{0,1}, ..., H_{0,r}, \pi_1^\perp H_{0,1}^{(1)}, ..., \pi_1^\perp H_{0,r}^{(1)}\},
\end{aligned}$$

for some meromorphic data $H_{0,1}, ..., H_{0,r}$.
Assume now that F_0 is not trivial and choose meromorphic data $\{L_{0,i}\}_{1\leq i\leq r}$ in F_0 and $\{E_{0,j}\}_{1\leq j\leq l}$ in $F_0^\perp$ to span $\underline{\alpha}_1$. From Theorem 1.2 we know that

$$\begin{aligned}
\underline{\alpha}_1 &= \operatorname{span}\{L_{0,i}, E_{0,j}\}_{(1\leq i\leq r,\ 1\leq j\leq l)}\\
\underline{\alpha}_2 &= \operatorname{span}\{L_{0,i} + \pi_1^\perp H_{1,i}, E_{0,j} + \pi_1^\perp H_{1,r+j}, \pi_1^\perp L_{0,i}^{(1)}, \pi_1^\perp E_{0,j}^{(1)}\}_{1\leq i\leq r,\ 1\leq j\leq l},
\end{aligned}$$

where the $\{H_{1,s}\}_{1\leq s\leq r+l}$ are $\mathbb{C}^n$-valued meromorphic functions.
It is easily seen that if the $H_{1,i}$ $(1 \leq i \leq r)$ lie in $F_0^\perp$ and the $H_{1,r+j}$ $(1 \leq j \leq l)$ lie in F_0 then $\phi_1 = (\pi_{F_0} - \pi_{F_0^\perp})(\pi_1 - \pi_1^\perp)$ commutes with π_2. As we shall see, eventually rearranging indexes, α_2 must be given this way.
Notice that, in the decomposition of $\underline{F}_2$ given by (2.2), $\underline{\alpha}_2 \cap F_1$ is a holomorphic subbundle of $(\mathbb{C}^n, D_{\bar{z}}^{\phi_1})$, since it is spanned by the sections $\{L_{0,i} + \pi_1^\perp H_{1,i}\}$ and $\{E_{0,j} + \pi_1^\perp H_{1,r+j}\}$ $(1 \leq i \leq r,\ 1 \leq j \leq l)$, which are holomorphic sections of that bundle. As we shall see later on, $\underline{\alpha}_2^\perp \cap F_1^\perp$ is a anti-holomorphic subbundle of the same bundle.

One of the main ingredients to develop the theory when dealing with harmonic maps into $G_*(\mathbb{C}^n)$ is the following result, already suggested by the previous examples.

Proposition 2.3. *Let $\phi : M^2 \to G_*(\mathbb{C}^n)$ be a harmonic map and $\underline{\alpha}$ a uniton for ϕ. Then, the harmonic map $\tilde{\phi} = \phi(\pi_\alpha - \pi_\alpha^\perp)$ is $G_*(\mathbb{C}^n)$-valued if, and only if, ϕ splits $\underline{\alpha}$. In that case, $\tilde{\phi} = \phi \cap \underline{\alpha} \oplus \phi^\perp \cap \underline{\alpha}^\perp$, where $\phi \cap \underline{\alpha}$ and $\phi^\perp \cap \underline{\alpha}^\perp$ are, respectively, holomorphic and anti-holomorphic subbundles of $(\mathbb{C}^n, D_{\bar{z}}^\phi)$.*

In the case of harmonic maps $\phi : M^2 \to \mathbf{U}(n)$, the holomorphic data $H_{i,j}$ of Theorem 1.2 could be freely chosen. We may inquire which conditions we must impose to the $H_{i,j}$ to get $\phi(M^2) \subseteq G_k(\mathbb{C}^n)$. The preceding proposition indicates that the splitting idea must be present in the initial data in order to obtain Grassmannian-valued harmonic maps.

Definition 2.4. Let F_0 be a constant subspace in $\mathbb{C}^n$. An $r \times n$ F_0-*array* is a family of meromorphic $\mathbb{C}^n$-valued functions, $(K_{i,j})_{0\leq i\leq r-1, 1\leq j\leq n}$ such that, for each j, either

$$\begin{aligned} &\pi_{F_0^\perp}(K_{2k,j}) = 0 \text{ and } \pi_{F_0}(K_{2k+1,j}) = 0, \text{ for all } 0 \leq k \leq \tfrac{r-1}{2} \text{ or} \\ &\pi_{F_0}(K_{2k,j}) = 0 \text{ and } \pi_{F_0^\perp}(K_{2k+1,j}) = 0, \text{ for all } 0 \leq k \leq \tfrac{r-1}{2}. \end{aligned} \tag{2.3}$$

Theorem 2.5. *Let F_0 be a constant subspace in $\mathbb{C}^n$, $r \in \{0, 1, ..., n-1\}$ and $(K_{i,j})_{0\leq i\leq r-1,1\leq j\leq n}$ be an $r \times n$ F_0-array of $\mathbb{C}^n$-valued meromorphic functions on M^2. For each j, consider the meromorphic functions*

$$\begin{aligned} &H_{0,j} = K_{0,j} \text{ and} \\ &H_{i,j} = \sum_{s=1}^{i} (-1)^{s+i} \binom{i-1}{s-1} K_{s,j},\ i \geq 1. \end{aligned} \tag{2.4}$$

For each $0 \leq i \leq r-1$, set $\underline{\alpha}_{i+1}$ equal to the subbundle of $\mathbb{C}^n$ spanned by the vectors

$$\alpha_{i+1,j}^{(k)} = \sum_{s=k}^{i} C_s^i H_{s-k,j}^{(k)},\ (j = 1, ..., n,\ k = 0, ..., i).$$

Then, the map $\phi : M^2 \to U(n)$ defined by

$$\phi = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)...(\pi_r - \pi_r^\perp)$$

is harmonic.
Further, all harmonic maps from M^2 to $G_*(\mathbb{C}^n)$ of finite uniton number, and so harmonic maps from S^2 to $G_*(\mathbb{C}^n)$, are obtained this way.

From now on we will represent the meromorphic data K, by L, when it takes values in F_0, or by E if it take values in $F_0^\perp$.

Example 2.6. For a general n, let F_0 be a two dimensional subspace, $r = 3$ and $j = 2$. Let $L_{i,1} \in F_0$, $E_{i,1} \in F_0^\perp$ $(0 \leq i \leq 2)$ and consider the F_0-array

$$\begin{bmatrix} L_{0,1} & E_{0,1} \\ E_{1,1} & L_{1,1} \\ L_{2,1} & E_{2,1} \end{bmatrix}.$$

Then, using (2.4), one gets $H_{0,1} = L_{0,1}$, $H_{0,2} = E_{0,1}$, $H_{1,1} = E_{1,1}$, $H_{1,2} = L_{1,1}$, $H_{2,1} = -E_{1,1} + L_{2,1}$, and $H_{2,2} = -L_{1,1} + E_{2,1}$.
We will assume that $L_{0,1}$, $L_{0,1}^{(1)}$ are linearly independent and that $E_{0,1}$, $E_{0,1}^{(1)}$, $E_{0,1}^{(2)}$ are also linearly independent.
Therefore, the map $\phi = (\pi_{F_0} - \pi_{F_0}^\perp)...(\pi_3 - \pi_3^\perp)$ is harmonic and $G_*(\mathbb{C}^n)$-valued, where

$$\begin{aligned} \underline{\alpha}_1 &= \operatorname{span}\{\, L_{0,1}, E_{0,2}\} \\ \underline{\alpha}_2 &= \operatorname{span}\{\, L_{0,1} + \pi_1^\perp E_{1,1}, E_{0,1} + \pi_1^\perp L_{1,1}, \pi_1^\perp L_{0,1}^{(1)}, \pi_1^\perp E_{0,1}^{(1)}\} \\ \underline{\alpha}_3 &= \operatorname{span}\{\, L_{0,1} + \pi_1^\perp E_{1,1} + \pi_2^\perp \pi_1 E_{1,1}, E_{0,1} + \pi_1^\perp L_{1,1} + \pi_2^\perp \pi_1 L_{1,1} + \pi_2^\perp \pi_1^\perp E_{2,1}, \\ &\qquad (\pi_1^\perp + \pi_2^\perp) L_{0,1}^{(1)} + \pi_2^\perp \pi_1^\perp E_{1,1}^{(1)}, (\pi_1^\perp + \pi_2^\perp) E_{0,1}^{(1)} + \pi_2^\perp \pi_1^\perp E_{1,1}^{(1)}, \pi_2^\perp \pi_1^\perp E_{0,1}^{(2)}\} \end{aligned}$$

We remark that $\pi_2^\perp\pi_1^\perp L_{2,1}$ and $\pi_2^\perp\pi_1^\perp L_{0,1}^{(2)}$ vanish, since $L_{0,1}$ and $\pi_1^\perp H_{0,1}^{(1)}$ span F_0. It is easily seen from the decomposition $\underline{F}_1 = \underline{\alpha}_1 \cap F_0 \oplus \underline{\alpha}_1^\perp \cap F_0^\perp$ that the rank of the bundle F_1 is $n-2$; in fact $\operatorname{rank}(\underline{\alpha}_1 \cap F_0) = 1$ and $\operatorname{rank}(\underline{\alpha}_1^\perp \cap F_0^\perp) = n-3$.
In the same way we conclude that $\operatorname{rank}(\underline{F}_2) = 2$, since $\underline{\alpha}_2^\perp \cap \underline{F}_1^\perp = \{0\}$. Then from the decomposition $\underline{F}_3 = \underline{\alpha}_3 \cap \underline{F}_2 \oplus \underline{\alpha}_3^\perp \cap \underline{F}_2^\perp$ we obtain $\operatorname{rank}(\underline{F}_3) = n-3$ and ϕ is a harmonic map into $G_{n-3}(\mathbb{C}^n)$ with uniton number 3.

From now on, for a harmonic map $\phi : M^2 \to G_*(\mathbb{C}^n)$, we will represent by $\underline{F}_\phi$ the corresponding tautological bundle and for $\phi = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_r - \pi_r^\perp)$ we will write $\underline{F}_{\phi_i} = \underline{F}_i$, where $\phi_i = (\pi_0 - \pi_0^\perp)...(\pi_i - \pi_i^\perp)$ $(0 \leq i \leq r)$ and F_0 is a constante subspace of $\mathbb{C}^n$.
We let $h : G_k(\mathbb{C}^n) \to G_{n-k}(\mathbb{C}^n)$ represent the isometry given by $h(F) = F^\perp$. Of course, $\phi = \pi_{F_i} - \pi_{F_i}^\perp$ implies that $h(\phi) = \pi_{F_i}^\perp - \pi_{F_i}$. Hence,

Proposition 2.7. *If* $\phi_i = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_i - \pi_i^\perp)$, *then* $h \circ \phi_i = (\pi_0^\perp - \pi_0)(\pi_1 - \pi_1^\perp)...(\pi_i - \pi_i^\perp)$.

2.2. **Harmonic maps into $G_p(\mathbb{C}^n)$.**
Given a subspace F_0 of $\mathbb{C}^n$ the main ingredient in building harmonic maps of finite uniton number is the selection of meromorphic data with values in F_0 and $F_0^\perp$. Let k denote the dimension of the complex subspace F_0 of $\mathbb{C}^n$, r the uniton number and fix $0 \leq i \leq r-1$.
For each family $\{L_{i,j}\}_{1\leq j\leq n}$ such that $L_{a,j} = 0$ whenever $0 \leq a < i$ we use the notation: $l_i^t = \operatorname{rank}\operatorname{span}\{C_{i+t}^{i+t} L_{i,j}^{(t)}\}_{1\leq j\leq n}$, where $0 \leq t \leq r-i-1$.
Analogously, for each family $\{E_{i,j}\}_{1\leq j\leq n}$ such that $E_{a,j} = 0$ whenever $0 \leq a < i$ we use the notation $s_i^t = \operatorname{rank}\operatorname{span}\{C_{i+t}^{i+t} E_{i,j}^{(t)}\}_{1\leq j\leq n}$, where $0 \leq t \leq r-i-1$.
In this way we get two triangular $r \times r$ matrices

$$L = \begin{bmatrix} l_0^0 & 0 & \cdots & 0 \\ l_0^1 & l_1^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ l_0^{r-1} & l_1^{r-2} & \cdots & l_{r-1}^0 \end{bmatrix} \text{ and } S = \begin{bmatrix} s_0^0 & 0 & \cdots & 0 \\ s_0^1 & s_1^0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ s_0^{r-1} & s_1^{r-2} & \cdots & s_{r-1}^0 \end{bmatrix},$$

where, in each column $i+1$, the entries $(l_i^0, \cdots l_i^{r-i-1})$ and $(s_i^0, \cdots s_i^{r-i-1})$ are decreasing sequences.
Notice that the sum of all entries of both matrices up to the i'th line is exactly the rank of α_{i+1}. Of course, the sum of all entries of L has to be less or equal than k, the sum of all entries of S has to be less or equal than $n-k$ and the sum of all entries of both matrices has to be less or equal than $n-1$.
Under the above conditions we will say that *the pair* (L, S) *is adapted to* F_0. From now on, F_0 will denote a subspace of $\mathbb{C}^n$ with dimension k and (L, S) will represent an adapted pair of matrices of order r.

Example 2.8. Let $n = 10$, $k = 5$ and consider an F_0 array of the form

$$\begin{bmatrix} L_{0,1} & E_{0,1} & 0 & 0 & 0 \\ E_{1,1} & L_{1,1} & E_{1,2} & L_{1,2} & 0 \\ L_{2,1} & E_{2,1} & L_{2,2} & E_{2,2} & L_{2,3} \end{bmatrix}$$

to build a uniton number 3 harmonic map $\varphi : S^2 \to G_*(\mathbb{C}^{10})$, according to Theorem 2.5. We know that $\underline{\alpha}_1 = \text{span}\,\{L_{0,1}, E_{0,1}\}$, and

$$\begin{aligned}
\underline{\alpha}_2^{(0)} &= \text{span}\{\, L_{0,1} + C_1^1 E_{1,1}, E_{0,1} + C_1^1 L_{1,1}, C_1^1 L_{1,2}, C_1^1 E_{1,2}\},\\
\underline{\alpha}_2^{(1)} &= \text{span}\{\, C_1^1 L_{0,1}^{(1)}, C_1^1 E_{0,1}^{(1)}\},
\end{aligned}$$

$$\begin{aligned}
\underline{\alpha}_3^{(0)} &= \text{span}\{\, L_{0,1} + C_1^2 E_{1,1} + C_2^2 L_{2,1}, E_{0,1} + C_1^2 L_{1,1} + C_2^2 E_{2,1},\\
&\qquad C_1^2 L_{1,2} + C_2^2 E_{2,2}, C_1^1 E_{1,2} + C_2^2 L_{2,2}, C_2^2 L_{2,3}\},\\
\underline{\alpha}_3^{(1)} &= \text{span}\{\, C_1^2 L_{0,1}^{(1)} + C_2^2 E_{1,1}^{(1)}, C_1^2 E_{0,1}^{(1)} + C_2^2 L_{1,1}^{(1)}, C_2^2 E_{1,2}^{(1)}\},\\
\underline{\alpha}_3^{(2)} &= \text{span}\{\, C_2^2 L_{0,1}^{(2)}\},
\end{aligned}$$

where we have assumed that $C_2^2 L_{1,2}^{(1)} = C_2^2 E_{0,1}^{(2)} = 0$, $\text{rank}(\underline{\alpha}_1) = 2$, $\text{rank}(\underline{\alpha}_2) = 6$ and $\text{rank}(\underline{\alpha}_3) = 9$.

As we have seen before, underlying the construction of a uniton three harmonic map there is a pair (L, S) of 3×3 diagonal matrices adapted to F_0, say

$$L = \begin{bmatrix} l_0^0 & 0 & 0 \\ l_0^1 & l_1^0 & 0 \\ l_0^2 & l_1^1 & l_2^0 \end{bmatrix} \text{ and } S = \begin{bmatrix} s_0^0 & 0 & 0 \\ s_0^1 & s_1^0 & 0 \\ s_0^2 & s_1^1 & s_2^0 \end{bmatrix}.$$

We remark that, for each $j \in \{0, 1, 2\}$ and $i \leq j$, $\text{rank}(\underline{\alpha}_{j+1}^i) = \sum_{k=0}^{j-i}(l_k^i + s_k^i)$, so that the rank of $\underline{\alpha}_{j+1}$ is $\sum_{i=0}^{j}\sum_{k=0}^{j-i}(l_k^i + s_k^i)$.
Since $C_2^2 L_{1,2}^{(1)} = C_2^2 E_{0,1}^{(2)} = 0$, in the particular case of this example we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In the sequel, given F_0 and an adapted pair (L, S) of $r \times r$ matrices, we will use the following notation: $A_0 = B_0 = 0$ and, for each $i \in \{1, ..., r-1\}$, $A_i = \sum_{r=0}^{i-1} l_r^0 + s_r^0$ and $B_i = A_i + l_i^0$.

Definition 2.9. An $r \times n$ F_0-array $(K_{i,j})_{0 \leq i \leq r-1, 1 \leq j \leq n}$ is said to match the the ordered pair (L, S) if, for each $i \in \{0, ..., r-1\}$, the following conditions hold:
(i) $\pi_{F_0^\perp}(K_{i,j}) = 0$, $\forall\, A_i + 1 \leq j \leq A_i + l_i^0$ and $\pi_{F_0}(K_{i,j}) = 0$, $\forall\, B_i + 1 \leq j \leq B_i + s_i^0 = A_{i+1}$.
(ii) For each $0 \leq j \leq i$, $\text{rank span}\left\{C_i^i K_{j,A_j+1}^{(i-j)}, ..., C_i^i K_{j,A_j+l_j^0}^{(i-j)}\right\} = l_j^{i-j}$ and
$\text{rank span}\left\{C_i^i K_{j,B_j+1}^{(i-j)}, ..., C_i^i K_{j,B_j+s_j^0}^{(i-j)}\right\} = s_j^{i-j}$.
(iii) $\text{rank span}\left\{C_i^i K_{j,A_j+1}^{(i-j)}, ..., C_i^i K_{j,A_j+l_j^0}^{(i-j)}\right\}_{0 \leq j \leq i} = \sum_{j=0}^{i} l_j^{i-j}$ and

$$\text{rank span}\left\{C_i^i K_{j,B_j+1}^{(i-j)}, ..., C_i^i K_{j,B_j+s_j^0}^{(i-j)}\right\}_{0 \leq j \leq i} = \sum_{j=0}^{i} s_j^{i-j}.$$

Remark 2.10. (i) We easily conclude that the rank of the bundle $\underline{\alpha}_{i+1}$ is $\sum_{t=0}^{i}\sum_{j=0}^{i-t}(l_t^j + s_t^j)$, the sum of all entries of the first i lines of both triangular matrices.
(ii) The l_i^t and s_i^t are independent of the choice of the complex coordinate z; in fact, once $\alpha_{j+1}^{(0)}$ is defined, letting

$$V_j = \operatorname{span}\{C_j^j K_{j,A_j+1}^{(i-j)}, ..., C_j^j K_{j,A_j+l_j^0}^{(i-j)}\} = \begin{cases} \ker \pi_j|_{\alpha_{j+1}^{(0)}} \cap F_{j+1}^{\perp}, & \text{if } j \text{ odd} \\ \ker \pi_j|_{\alpha_{j+1}^{(0)}} \cap F_{j+1}, & \text{if } j \text{ even}, \end{cases}$$

we have $l_j^0 = \operatorname{rank} V_j$ and $l_j^{i-j} = \operatorname{rank} A_z^{\phi_i} ... A_z^{\phi_{j+1}} V_j$ $(i \geq j)$. Analogously with respect to the s_i^t.

An induction argument allows the following result:

Theorem 2.11. *Let $r \in \{1, ..., n-1)$, F_0 be a k-dimensional subspace of $\mathbb{C}^n$ and consider a pair (L, S) adapted to F_0. For any F_0-array $(K_{i,j})_{0\leq i\leq r-1, 1\leq j\leq n}$ which matches (L, S) and $i \in \{0, ..., r\}$, the rank of the tautological bundle $\underline{F}_i$ corresponding to the harmonic map $\phi_i = (\pi_0 - \pi_0^{\perp})...(\pi_i - \pi_i^{\perp})$ is given by*

$$\begin{cases} k + \displaystyle\sum_{j=0}^{\frac{i}{2}-1}\sum_{t=0}^{2j+1}(s_t^{2j+1-t} - l_t^{2j+1-t}) & \text{if } i \text{ is even} \\ n - \left[k + \displaystyle\sum_{j=0}^{\frac{i-1}{2}}\sum_{t=0}^{2j}(s_t^{2j-t} - l_t^{2j-t})\right] & \text{if } i \text{ is odd,} \end{cases} \tag{2.5}$$

Using Theorem 2.11 we can see that, when we start with a harmonic map $\phi : M^2 \to G_p(\mathbb{C}^n)$ and add a uniton $\underline{\alpha}$, the harmonic map $\phi(\pi_{\underline{\alpha}} - \pi_{\underline{\alpha}^{\perp}})$ does not, in general, take values in the same Grassmannian. However, in certain cases, it is possible to add unitons in such a way that the successive harmonic maps stay in the same Grassmannian (see Example 2.12).

Example 2.12. Let us consider $G_4(\mathbb{C}^8)$ as target manifold and start with a 4-dimensional complex subspace F_0 of $\mathbb{C}^8$. We select the ordered pair (L, S) adapted to F_0 with

$$L = S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and take a F_0-array which matches the pair (L, S). Then the harmonic maps ϕ_1, ϕ_2 and ϕ_3 all have values in $G_4(\mathbb{C}^8)$, as it is easily seen from Theorem 2.11, since $l_0^j = s_0^j$ for every $j \in \{0, 1, 2\}$.

Example 2.13. In this example, using Theorem 2.11, we will describe all harmonic maps $\phi : S^2 \to G_2(\mathbb{C}^5)$ with uniton number 3. Let F_0 be a k- dimensional complex subspace of $\mathbb{C}^5$ $(0 \leq k \leq 5)$ and (L, S) a pair adapted to F_0,

$$L = \begin{bmatrix} l_0^0 & 0 & 0 \\ l_0^1 & l_1^0 & 0 \\ l_0^2 & l_1^1 & l_2^0 \end{bmatrix}, \quad S = \begin{bmatrix} s_0^0 & 0 & 0 \\ s_0^1 & s_1^0 & 0 \\ s_0^2 & s_1^1 & s_2^0 \end{bmatrix}.$$

The sum of all entries of both matrices has to be less or equal than 4 and the uniton number three condition implies that at least one element of the third lines of the matrices has to be different from zero. From Theorem 2.11 we know that

$$2 = 5 - [k + (s_0^0 - l_0^0) + (s_0^2 - l_0^2) + (s_1^1 - l_1^1) + (s_2^0 - l_2^0)]. \tag{2.6}$$

We have to analyze the different cases according to the dimension of F_0. (a) Considering $k = 5$, i.e, $S = 0$, we have $2 = l_0^0 + l_0^2 + l_1^1 + l_2^0$. The only possibility is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

since $\underline{\alpha}_1$ is full. This gives rise to the unitons

$$\begin{aligned}
&\underline{\alpha}_1 = \operatorname{span}\{L_{0,1}\}, \\
&\underline{\alpha}_2 = \operatorname{span}\{L_{0,1}, \pi_1^\perp L_{0,1}^{(1)}\} \text{ and} \\
&\underline{\alpha}_3 = \operatorname{span}\{L_{0,1}, \pi_1^\perp L_{0,1}^{(1)}, \pi_2^\perp \pi_1^\perp L_{0,1}^{(2)}\}.
\end{aligned}$$

(b) Now we analise the case $k = 4$. Here we have $1 = (l_0^0 - s_0^0) + (l_0^2 - s_0^2) + (l_1^1 - s_1^1) + (l_2^0 - s_2^0)$. It is not hard to check that the only possibility is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore we choose our meromorphic data $L_{0,1}$, $L_{1,1}$ and $L_{2,1}$ with values in F_0 and $E_{0,1}$ with values in $F_0^\perp$. This corresponds to

$$\begin{aligned}
&\underline{\alpha}_1 = \operatorname{span}\{L_{0,1}, E_{0,1}\}, \\
&\underline{\alpha}_2 = \operatorname{span}\{L_{0,1}, E_{0,1} + \pi_1^\perp L_{1,1}, \pi_1^\perp L_{0,1}^{(1)}\} \text{ and} \\
&\underline{\alpha}_3 = \operatorname{span}\{L_{0,1} + \pi_2^\perp \pi_1^\perp L_{2,1}, E_{0,1} + C_1^2 L_{1,1}, \pi_1^\perp L_{0,1}^{(1)}, \pi_2^\perp \pi_1^\perp L_{0,1}^{(2)}\}.
\end{aligned}$$

(c) Consider $k = 0$, which corresponds to $L = 0$ and implies $3 = s_0^0 + s_0^2 + s_1^1 + s_2^0$. We remark that cases like

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ or } S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

although satisfy our equation, have to be excluded, since do not fulfil the fullness of $\underline{\alpha}_1$. Hence the only possibility is

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and we choose our meromorphic data $E_{0,1}$, $E_{2,1}$ and $E_{2,2}$ with values in $F_0^\perp$ to get the harmonic map $\phi = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)(\pi_2 - \pi_2^\perp)(\pi_3 - \pi_3^\perp)$, where

$$\underline{\alpha}_1 = \operatorname{span}\{E_{0,1}\},$$
$$\underline{\alpha}_2 = \operatorname{span}\{E_{0,1}, \pi_1^\perp E_{0,1}^{(1)}\} \text{ and}$$
$$\underline{\alpha}_3 = \operatorname{span}\{E_{0,1} + \pi_2^\perp \pi_1^\perp E_{2,1}, \pi_1^\perp E_{0,1}^{(1)}, \pi_2^\perp \pi_1^\perp E_{0,1}^{(2)}, \pi_2^\perp \pi_1^\perp E_{2,2}\}.$$

The cases $k = 1, 2, 3$ must be excluded. Regarding $k = 2, 3$, the fullness of $\underline{\alpha}_1$ would imply that

$$L_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } S_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

in the first case and

$$L_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } S_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

in the second case, which is not adequate for $G_2(\mathbb{C}^5)$ as the sum of these entries is 5. As for the case $k = 1$, the fullness of $\underline{\alpha}_1$ would require

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which does not satisfy (2.6). Hence, the three cases (a), (b) and (c) are the only ones yielding uniton number three harmonic maps into $G_2(\mathbb{C}^5)$.
From Proposition 2.7 interchanging $F_0^\perp$ with F_0 and S with L, we get the description of all uniton number three harmonic maps into $G_3(\mathbb{C}^5)$.

It is known that, for a harmonic map $\phi : M^2 \to G_p(\mathbb{C}^n)$, the maximal uniton number is less or equal than 2min $\{p, n-p\}$ [7, 1]. We will see, later on, that this estimate is sharp only when $n \neq 2p$ and F_0 is trivial.
In the next theorem, fixing a subspace F_0 with dimension k, we present an estimate for the uniton number of a harmonic map $\phi = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_i - \pi_i^\perp)$, when $2p \leq n$. This estimate is sharp and covers all situations, for if $n > p$ and $\phi : M^2 \to G_p(\mathbb{C}^n)$ is harmonic, $h \circ \phi : M^2 \to G_{n-p}(C^n)$ is a harmonic map with the same uniton number and $2(n-p) < n$.

Theorem 2.14. *Let F_0 be a k-dimensional complex subspace of $\mathbb{C}^n$ and $\phi = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_{r_k} - \pi_{r_k}^\perp)$ be a harmonic map into $G_p(\mathbb{C}^n)$, where r_k is the uniton number and $2p \leq n$. Then,*
(i) $r_k \leq \min\{2p - k - a_k, n-1\}$if $k < p$;
(ii) $r_k \leq p$ if $k \geq p$ and $k + p \leq n$;
(iii) $r_k \leq 2p - (n-k) - a_k$ if $k \geq p$ and $k + p > n$,
where $a_k = \begin{cases} 1 \text{ if } k \text{ is even and } k < p \text{ or } n-k \text{ is even and } k \geq p \\ 0 \text{ if } k \text{ is odd and } k < p \text{ or } n-k \text{ is odd and } k \geq p. \end{cases}$
Moreover, the above estimates are sharp, except in the case $k = p$, where $r_k \leq p - 1$.

A glance at the list of possibilities given by the previous proposition allows to verify that the maximal uniton number is realized when $k = 0$ and $2p \leq n$, or when $k = n$ and $2p \geq n$.

Example 2.15. Assume $\phi = (\pi_0 - \pi_0^\perp)...(\pi_r - \pi_r^\perp) : S^2 \to G_2(\mathbb{C}^4)$, where $k = 2$. From Theorem 2.14 we know that, for $k = 2$ fixed, the maximal uniton number is 2. Let us describe those harmonic maps. Consider an adapted pair (L, S) of ordered 2×2 diagonal matrices adapted to F_0. From (2.5), we get
$2 = 2 + \sum_{j=0}^{1}(s_j^{1-j} - l_j^{1-j})$ or $s_o^1 + s_1^0 = l_0^1 + l_1^1$.
Clearly the only possibility is

$$L = S = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus we see that we shall start with meromorphic data L_0 and E_0 with values in F_0 and $F_0^\perp$, respectively, to build the unitons

$$\underline{\alpha}_1 = \mathrm{span}\{L_0, E_0\} \text{ and}$$
$$\underline{\alpha}_2 = \mathrm{span}\{L_0, E_0, \pi_1^\perp L_0^{(1)}, \pi_1^\perp E_0^{(1)}\}.$$

Example 2.16. Let us now describe the construction of all harmonic maps $\phi : S^2 \to G_3(\mathbb{C}^8)$ (respectively $\phi : S^2 \to G_5(\mathbb{C}^8)$) of the type $\phi = (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_r - \pi_r^\perp)$, where $k = 4$ and r_k is maximal.
We know from Theorem 2.14 that $r_k = 3$. Hence, using Theorem 2.11, we get $1 = (l_0^0 + l_0^2 + l_1^1 + l_2^0) - (s_0^0 + s_0^2 + s_1^1 + s_2^0)$.
Let us try to describe the possible pairs (L, S) of diagonal 3×3 matrices adapted to F_0. As above, since α_1 is full, we must have $l_0^i \neq 0$ and $s_0^i \neq 0$ for every $i \in \{1, 2, 3\}$. It is easily seen that the only possibility is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore we must choose our meromorphic data, $\{L_{0,1}, L_{1,1}, L_{2,1}, L_{2,2}\}$ with values in F_0, and $\{E_{0,1}, E_{1,1}, E_{2,1}\}$ with values in $F_0^\perp$. This gives rise to the unitons

$$\begin{aligned} \alpha_1 &= \mathrm{span}\{L_{0,1}, E_{0,1}\}, \\ \alpha_2 &= \mathrm{span}\{L_{0,1} + C_1^1 E_1^1, E_{0,1} + C_1^1 L_{1,1}, C_1^1 L_{0,1}^{(1)}, C_1^1 E_{0,1}^{(1)}\}, \\ \alpha_3 &= \mathrm{span}\{L_{0,1} + C_2^1 E_1^1 + C_2^2 L_{2,1}, E_{0,1} + C_1^1 L_{1,1} + C_2^2 E_{2,1}\}, \\ &\qquad C_2^1 L_{0,1}^{(1)} + C_2^2 E_{1,1}^{(1)}, C_1^1 E_{0,1}^{(1)} + C_2^2 L_{1,1}^{(1)}, C_2^2 L_{2,2}, C_2^2 L_{0,1}^{(2)}, C_2^2 E_{0,1}^{(2)}\}. \end{aligned}$$

Interchanging L and S and choosing the same holomorphic data we get the description of all harmonic maps $\phi : S^2 \to C_5(\mathbb{C}^8)$ with uniton number 3.

We may synthesize the above results in the following statement concerning harmonic maps $\phi : S^2 \to G_p(\mathbb{C}^n)$ into a fixed Grassmannian.

Theorem 2.17. *Let $q = \min\{p, n-p\}$, $k \in \{0, ..., n\}$ and r_k be under the conditions of Theorem 2.14. Taking a pair (L, S) of $i \times i$ matrices ($1 \leq i \leq r_k$) adapted to F_0, whose entries satisfy*

equations (2.5)*, and an array* $(K_{i,j})$ *matching* (L, S)*, the map*

$$\phi_i = \begin{cases} (\pi_0 - \pi_0^\perp)(\pi_1 - \pi_1^\perp)...(\pi_i - \pi_i^\perp), \text{ if } q = p \\ (\pi_0^\perp - \pi_0)(\pi_1 - \pi_1^\perp)...(\pi_i - \pi_i^\perp), \text{ if } q = n - p \end{cases}$$

is a harmonic map into $G_p(\mathbb{C}^n)$*. Moreover, all harmonic maps* $\phi : S^2 \to G_p(\mathbb{C}^n)$ *are obtained this way.*

2.3. **A note on the Grassmannian model.**
Let $\mathcal{H}$ denote the Hilbert space $L^2(S^1, \mathbb{C}^n)$ and let $\mathcal{H}_+$ denote the linear closure of elements of the form $\sum_{k\geq 0} \lambda^k e_j$ where $\{e_j\}_{1\leq j\leq n}$ form the standard basis of $\mathbb{C}^n$. The algebraic loop group $\Omega^{\mathrm{alg}}\mathbf{U}(n)$ consists of maps $\gamma : S^1 \to \mathbf{U}(n)$ with $\gamma(1) = I$ and such that $\gamma(\lambda) = \sum_{k=0}^r \lambda^k A_j$, for some integer r and $A_j \in \mathfrak{gl}(n, \mathbb{C})$. It acts naturally on $\mathcal{H}$ and the correspondence $\gamma \to \gamma(H_+)$ identifies $\Omega^{\mathrm{alg}}\mathbf{U}(n)$ with the *algebraic Grassmannian* consisting of all subspaces W of $\mathcal{H}$ such that $\lambda W \subseteq W$ and $\lambda^r\mathcal{H}_+ \subseteq W \subseteq \mathcal{H}_+$ for some r [12, 13]. In particular, we may identify W with the coset $W + \lambda^r\mathcal{H}_+$ in the finite-dimensional vector space $\mathcal{H}_+/\lambda^r\mathcal{H}_+$; this vector space is canonically identified with $\mathbb{C}^{rn}$ via the isomorphism

$$(R_0, R_1, ..., R_{r-1}) \to R_0 + \lambda R_1 + ... + \lambda^{r-1}R_{r-1} + \lambda^r\mathcal{H}_+. \tag{2.7}$$

Now, let $\phi : M^2 \to \mathbf{U}(n)$ be a harmonic map of uniton number at most r and Φ be its unique type one (polynomial) extended solution. We may naturally interpret Φ as a smooth map $\Phi : M^2 \to \Omega^{\mathrm{alg}}\mathbf{U}(n)$. With the above identifications, we then have a holomorphic map $W = \Phi(\mathcal{H}_+)$ from M^2 into the into $G_*(\mathbb{C}^{rn})$. Equivalently [8], a holomorphic subbundle $\underline{W}$ of the trivial bundle $M^2 \times \mathbb{C}^{rn}$ satisfying

$$\lambda\underline{W}_{(1)} \subseteq \underline{W}, \tag{2.8}$$

where $\underline{W}_{(i)}$ ($i \geq 0$) denotes the subbundle spanned by (local) sections of $\underline{W}$ and their first i derivatives with respect to any complex coordinate z on M^2. We call $\underline{W}$ the *Grassmannian model* of ϕ (or Φ).
All such subbundles $\underline{W}$ are given by taking an arbitrary holomorphic subbundle $\underline{X}$ of $\underline{\mathbb{C}}^{rn}$ and setting $\underline{W}$ equal to the coset [9]

$$\underline{W} = \underline{X} + \lambda\underline{X}_{(1)} + \lambda^2\underline{X}_{(2)} + \cdots + \lambda^{r-1}\underline{X}_{(r-1)}. \tag{2.9}$$

For any $i \geq 0$ and meromorphic vectors $(H_0, H_1, \ldots, H_i)$, set

$$R_i = \sum_{l=0}^{i} \binom{i}{l} H_l\,. \tag{2.10}$$

The isomorphism (2.7) allows us to describe the Grassmannian model of a finite uniton number harmonic map $\phi : M^2 \to \mathbf{U}(n)$ in the following way [3]:

Theorem 2.18. *Let* $r \geq 1$*, and let* $\underline{B}$ *and* $\underline{X}$ *be holomorphic subbundles of* $\mathbb{C}^{rn}$ *related by the linear isomorphism*

$$\underline{B} \ni H = (H_0, H_1 \ldots, H_{r-1}) \to R = (R_0, R_1, \ldots, R_{r-1}) \in \underline{X}$$

given by (2.10). *Write*

$$\underline{\alpha}_{i+1}^{(k)} = \{\sum_{s=k}^{i} C_s^i H_{s-k}^{(k)},\ H \in \Gamma_{\text{hol}}(\underline{B})\} \textit{ and } \underline{\alpha}_{i+1} = \sum_{k=0}^{i} \underline{\alpha}_{i+1}^{(k)}.$$

Let $\phi : M^2 \to \mathbf{U}(n)$ be the harmonic map given by (1.4) *and $W : M^2 \to G_*(\mathbb{C}^{rn})$ be the holomorphic map given by* (2.9)*. Then W is the Grassmannian model of ϕ.*

Let F_0 denote a constant subspace in $\mathbb{C}^n$. We say that a polynomial $R \in \mathcal{H}_+/\lambda^r\mathcal{H}_+$ is F_0*-adapted* if its coefficients have image alternately in F_0 and $F_0^\perp$, i.e., $L(\lambda) = \sum_{i=0}^{r-1} L_i\lambda^i$ and *either*
(i) R_i has image in F_0 for i even, and in $F_0^\perp$ for i odd, *or*
(ii) R_i has image in $F_0^\perp$ for i even, and in F_0 for i odd.
Note that when F_0 is trivial, a polynomial is F_0-adapted if and only if it is even or odd, i.e, has coefficients of all odd or all even powers of λ equal to zero. Using the Grassmannian model, we have the following characterization of maps into $G_*(\mathbb{C}^n)$ [3]:

Proposition 2.19. *Φ is the extended solution of a harmonic map into a Grassmannian if and only if W has a spanning set consisting of F_0-adapted polynomials, or, equivalently, W is given by* (2.9) *for some X which has a spanning set consisting of F_0-adapted polynomials.*

Now, let ϕ be a Grassmannian-valued harmonic map given as in Theorem 2.5 for some F_0-array $(K_{i,j})_{0\le i\le r-1, 1\le j\le n}$. Using (2.4) and (2.10) one can easily check that

$$R_{i,j} = \sum_{s=0}^{i} K_{s,j}.$$

Hence, the Grassmannian model for ϕ is given by

$$\underline{W} = \underline{X} + \lambda\underline{X}_{(1)} + \lambda^2\underline{X}_{(2)} + \cdots + \lambda^{r-1}\underline{X}_{(r-1)},$$

where $\underline{X} = (K_{0,j}, K_{0,j} + K_{1,j}, ..., K_{0,j} + ... + K_{r-1,j})$. Since

$$\begin{aligned} K_{0,j} + \lambda K_{1,j} + ... + \lambda^{r-1}K_{r-1,j} = {} & K_{0,j} + \lambda(K_{0,j} + K_{1,j}) + ... + \lambda^{r-1}(K_{0,j} + ... + K_{r-1,j}) \\ & -\lambda(K_{0,j} + \lambda(K_{0,j} + K_{1,j}) + ... + \lambda^{r-1}(K_{0,j} + ... + K_{r-1,j}), \end{aligned}$$

we can also write $\underline{W} = \underline{\tilde{X}} + \lambda\underline{\tilde{X}}_{(1)} + \lambda^2\underline{\tilde{X}}_{(2)} + \cdots + \lambda^{r-1}\underline{\tilde{X}}_{(r-1)}$, where $\underline{\tilde{X}} = (K_{0,j}, ..., K_{r-1,j})$: it is clear that $\underline{\tilde{X}}$ is spanned by F_0-adapted polynomials.

Hence, we now can easily construct explicitly our harmonic map ϕ from its Grassmannian model W: given a set $R_{i,j}$ of F_0-adapted polynomials that generate $\underline{W}$, we set $K_{i,j} = R_{i,j}$ and construct the map ϕ as in Theorem 2.5.

2.4. Proof of the main results.

Let $\phi : M^2 \to G_*(\mathbb{C}^n)$ be a harmonic map and $\underline{\alpha}$ a uniton for ϕ. From [14], we know that $\tilde{\phi} = \phi(\pi_\alpha - \pi_\alpha^\perp)$ lies in $G_*(\mathbb{C}^n)$ if and only if π_α and ϕ commute. This means that $\underline{F}_\phi$ splits the eigenspaces of π_α so that $\underline{\alpha} = \underline{\alpha} \cap \underline{F}_\phi \oplus \underline{\alpha} \cap \underline{F}_\phi^\perp$. As a consequence,

$$\tilde{\phi} = \pi_{\alpha\cap F_\phi \oplus \alpha^\perp \cap F_\phi^\perp} - \pi_{\alpha^\perp \cap F_\phi \oplus \alpha \cap F_\phi^\perp}.$$

Recall from [14] the following facts.

Proposition 2.20. *Let $\phi : M^2 \to G_*(\mathbb{C}^n)$ be a harmonic map of finite uniton number r. Then, there are unique proper unitons $\underline{\alpha}_1, \dots, \underline{\alpha}_r$ satisfying the covering condition (1.7) and with $\underline{\alpha}_1$ full and a constant map $Q = (\pi_{F_0} - \pi_{F_0}^\perp)$ such that*

$$\phi = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)\dots(\pi_r - \pi_r^\perp). \tag{2.11}$$

Moreover, each partial map $\phi_{r'} = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)\dots(\pi_{r'} - \pi_{r'}^\perp)$ maps into $G_*(\mathbb{C}^n)$ and commutes with $\pi_{r'+1}$.

In the sequel, we shall always consider a harmonic map $\phi : M^2 \to G_*(\mathbb{C}^n)$ factorized as in Proposition 2.20. Let ϕ be as in (2.11). We define recursively

$$\begin{aligned} &\underline{F}_1 = F_0 \cap \underline{\alpha}_1 \oplus F_0^\perp \cap \underline{\alpha}_1^\perp; \\ &\underline{F}_2 = \underline{F}_1 \cap \underline{\alpha}_2 \oplus \underline{F}_1^\perp \cap \underline{\alpha}_2^\perp; \\ &\quad\dots \\ &\underline{F}_r = \underline{F}_{r-1} \cap \underline{\alpha}_r \oplus \underline{F}_{r-1}^\perp \cap \underline{\alpha}_r^\perp. \end{aligned} \tag{2.12}$$

It is easy to check that ϕ lies in $G_*(\mathbb{C}^n)$ if and only if $\underline{F}_i$ decomposes $\underline{\alpha}_{i+1}$ for all $1 \leq i \leq r-1$. Moreover, in that case, each of the partial maps

$$\phi_{r'} = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)\dots(\pi_{r'} - \pi_{r'}^\perp)$$

also lies in $G_*(\mathbb{C}^n)$ and $\phi_{r'} = \pi_{\underline{F}_{r'}} - \pi_{\underline{F}_{r'}}^\perp$.

Corollary 2.21. *A_z^ϕ interchanges $\underline{F}_r$ and $\underline{F}_r^\perp$.*

Proof. Recall that $2A_z^\phi = \phi^{-1}\partial_z\phi = (\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)^{-1}\partial_z(\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)$. In particular, if f is a section of $\underline{F}_r$, we have that

$$\begin{aligned} 2A_z^\phi f &= (\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)\partial_z(\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)f = (\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)(\partial_z f - \pi_{\underline{F}_r}\partial_z f + \pi_{\underline{F}_r}^\perp \partial_z f) \\ &= 2(\pi_{\underline{F}_r} - \pi_{\underline{F}_r}^\perp)(\pi_{\underline{F}_r}^\perp \partial_z f) \in \underline{F}_r^\perp. \end{aligned}$$

If f is a section of $\underline{F}_r^\perp$, the argument is similar. □

Let S_j^i denote the sum of all ordered i-fold products of the form $\Pi_i \cdots \Pi_1$, where exactly j of the Π_l are $\pi_l^\perp$ and the other $i-j$ are π_l. For $i = 0$, set $S_j^i = I$, for $i < 0$ or $j > i > 0$, set $S_j^i = 0$. Then, [3]

$$S_j^i = \pi_i S_j^{i-1} + \pi_i^\perp S_{j-1}^{i-1}$$

and the S_j^i are related with the C_j^i by the formulae

$$C_k^i = \sum_{s=k}^{i} \binom{s}{k} S_s^i,$$

where $\binom{i}{s}$ denotes the binomial coefficient $i!/s!(i-s)!$.

Lemma 2.22. *Let $\underline{F}_i$, $1 \leq i \leq r$ be defined as in (2.12). Then, if $\underline{F}_i$ decomposes $\underline{\alpha}_{i+1}$ for all $1 \leq i \leq r-1$,*

$$\begin{array}{ll} S^i_j \pi_{F_0} A \in \underline{F}_i & \text{if } j \text{ even} \\ S^i_j \pi_{F_0} A \in \underline{F}_i^\perp & \text{if } j \text{ odd} \\ S^i_j \pi^\perp_{F_0} A \in \underline{F}_i^\perp & \text{if } j \text{ even} \\ S^i_j \pi^\perp_{F_0} A \in \underline{F}_i & \text{if } j \text{ odd}, \end{array} \tag{2.13}$$

for any A.

Proof. For the case $r=1$, assume that F_0 splits $\underline{\alpha}_1$. Let us show that (2.13) holds. As a matter of fact:

$$\begin{array}{ll} S^1_0 \pi_{F_0} A = \pi_1 \pi_{F_0} A \in \underline{F}_1 & (j \text{ even}) \\ S^1_1 \pi_{F_0} A = \pi_1^\perp \pi_{F_0} A \in \underline{F}_1^\perp & (j \text{ odd}) \\ S^1_0 \pi^\perp_{F_0} A = \pi_1 \pi^\perp_{F_0} A \in \underline{F}_1^\perp & (j \text{ even}) \\ S^1_1 \pi^\perp_{F_0} A = \pi_1^\perp \pi^\perp_{F_0} A \in \underline{F}_1 & (j \text{ odd}) \end{array}$$

Let us now establish the induction: assume the result holds up to r and that $\underline{F}_r$ splits $\underline{\alpha}_{r+1}$. Then,

$$S^{r+1}_j \pi_{F_0} A = \pi_{r+1} S^r_j \pi_{F_0} A + \pi^\perp_{r+1} S^r_{j-1} \pi_{F_0} A. \tag{2.14}$$

If j is odd, $S^r_j \pi_{F_0} A \in \underline{F}_r^\perp$. Since $j-1$ is even, $S^r_{j-1}\pi_{F_0} A \in \underline{F}_r$. Hence, (2.14) becomes $\pi_{r+1}\pi^\perp_{F_r} S^r_j \pi_{F_0} A + \pi^\perp_{r+1}\pi_{F_r} S^r_{j-1}\pi_{F_0} A \in \underline{F}_{r+1}$. The remaining cases have similar proofs. □

We know that a harmonic map is obtained as the product of unitons $\underline{\alpha}_i$ with $\underline{\alpha}_i$ given as in Theorem 1.2. To obtain maps into $G_*(\mathbb{C}^n)$, we must impose the following algebraic conditions on the meromorphic data $H_{i,j}$:

Proposition 2.23. *Let $(H_{i,j})_{0\leq i \leq r-1, 1 \leq j \leq n}$ be chosen in such a way that for all j, either $\pi_{F_0}(H_{0,j}) = 0$ and*

$$\begin{cases} \pi_{F_0}\left(\sum_{s=1}^{i} \binom{i-1}{s-1} H_{s,j}\right) = 0, & i \text{ even}, \\ \pi^\perp_{F_0}\left(\sum_{s=1}^{i} \binom{i-1}{s-1} H_{s,j}\right) = 0, & i \text{ odd}. \end{cases} \tag{2.15}$$

or $\pi^\perp_{F_0}(H_{0,j}) = 0$ and (2.15) holds, now with π_{F_0} replaced with $\pi^\perp_{F_0}$.
Then $\phi = (\pi_{F_0} - \pi^\perp_{F_0})(\pi_1 - \pi_1^\perp)...(\pi_r - \pi_r^\perp)$, with $\underline{\alpha}_i$ given as in (1.3), lies in $G_*(\mathbb{C}^n)$.
Moreover, for each $0 \leq i \leq r-1$, $\phi_i = \pi_{F_i} - \pi^\perp_{F_i}$ with:
(i) $\underline{F}_i \cap \underline{\alpha}_{i+1} \subseteq \underline{F}_{i+1}$ spanned by $\underline{\alpha}^{(k)}_{i+1,j}$, where j and k are such that k is even and $\pi^\perp_{F_0}(H_{0,j}) = 0$ or k is odd and $\pi_{F_0}(H_{0,j}) = 0$;
(ii) $\underline{F}_i^\perp \cap \underline{\alpha}_{i+1} \subseteq \underline{F}^\perp_{i+1}$ spanned by $\underline{\alpha}^{(k)}_{i+1,j}$, where j and k are such that k is odd and $\pi^\perp_{F_0}(H_{0,j}) = 0$ or k is even and $\pi_{F_0}(H_{0,j}) = 0$.

Proof. For $r=1$, it is trivial. For the case $r=2$, our initial data satisfies, for each j, $\pi_{F_0}(H_{0,j}) = 0$ and $\pi^\perp_{F_0}(H_{1,j}) = 0$ or $\pi^\perp_{F_0}(H_{0,j}) = 0$ and $\pi_{F_0}(H_{1,j}) = 0$. Moreover, $\underline{\alpha}_2$ is spanned by $\alpha^{(0)}_{2,j}$ and

$A_z^{\phi_1}(\alpha_{2,j}^{(0)})$. Now, $\alpha_{2,j}^{(0)}$ is either of the form $H_{0,j} + \pi_1^\perp H_{1,j}$ with $H_{0,j}$ in F_0 and $H_{1,j}$ in $F_0^\perp$ or $H_{0,j} + \pi_1^\perp H_{1,j}$ with $H_{0,j}$ in $F_0^\perp$ and $H_{1,j}$ in F_0. In the first case, $\alpha_{2,j}^{(0)}$ is a section of $\underline{F}_1$ (and of $\underline{\alpha}_2$ so that it is a section of $\underline{F}_2$) whereas in the second case we have a section of $\underline{F}_1^\perp$ (and hence of $\underline{F}_1^\perp \cap \underline{\alpha}_2 \subseteq \underline{F}_2^\perp$). Since $A_z^{\phi_1}$ interchanges $\underline{F}_1$ with $\underline{F}_1^\perp$, we conclude that $\underline{F}_1 \cap \underline{\alpha}_2$ is spanned by $\alpha_{2,j}^{(0)}$, for j such that $\pi_{F_0}^\perp(H_{0,j}) = 0$, and by $\alpha_{2,j}^{(1)} = -A_z^{\phi_1}(\alpha_{2,j}^{(0)})$, for j such that $\pi_{F_0}(H_{0,j}) = 0$.
Let us show the induction step: assume the result holds up to r. Without loss of generality, assume that j is such that $\pi_{F_0}^\perp H_{0,j} = 0$. Then, $\alpha_{r,j}^{(0)}$ lies in $\underline{F}_r$ and

$$\begin{aligned}
\alpha_{r+1,j}^{(0)} &= \alpha_{r,j}^{(0)} + \pi_r^\perp\Big(\sum_{t=0}^{r-1} C_t^{r-1} H_{t+1,j}\Big)\\
&= \alpha_{r,j}^{(0)} + \pi_r^\perp\Big(\sum_{t=0}^{r-1} S_t^{r-1}(\pi_{F_0} + \pi_{F_0}^\perp)\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big)\\
&= \alpha_{r,j}^{(0)} + \pi_r^\perp\Big(\sum_{\substack{t=0\\ t \text{ odd}}}^{r-1} S_t^{r-1}\pi_{F_0}\sum_{s=1}^{t+1}\binom{t}{s-1} H_{s,j} + \sum_{\substack{t=0\\ t \text{ even}}}^{r-1} S_t^{r-1}\pi_{F_0}^\perp\sum_{s=1}^{t+1}\binom{t}{s-1} H_{s,j}\Big)\\
&+\pi_r^\perp\Big(\sum_{\substack{t=0\\ t \text{ odd}}}^{r-1} S_t^{r-1}\pi_{F_0}^\perp\sum_{s=1}^{t+1}\binom{t}{s-1} H_{s,j} + \sum_{\substack{t=0\\ t \text{ even}}}^{r-1} S_t^{r-1}\pi_{F_0}\sum_{s=1}^{t+1}\binom{t}{s-1} H_{s,j}\Big).
\end{aligned}$$

Using Lemma 2.22, the first two terms lie in $\underline{F}_r$ whereas the last vanishes from our hypothesis. Hence $\alpha_{r+1,j}^{(0)} \in \underline{F}_r \cap \underline{\alpha}_{r+1} \subseteq \underline{F}_{r+1}$. Since $\alpha_{r+1,j}^{(k)} = -A_z^{\phi_r}(\alpha_{r+1,j}^{(k-1)})$ and $A_z^{\phi_r}$ interchanges $\underline{F}_r$ and $\underline{F}_r^\perp$ the conclusion now easily follows. □

Proof of Theorem 2.5. Let F_0 be a constant subspace of $\mathbb{C}^n$ and $(K_{i,j})_{0\le i\le r-1, 1\le j\le n}$ denote a F_0-array. Let $H_{i,j}$ be defined as in (2.4). It is easily seen that these equations are equivalent to

$$\begin{aligned}
H_{0,j} &= K_{0,j} \text{ and}\\
K_{i,j} &= \sum_{s=1}^{i}\binom{i-1}{s-1} H_{s,j}.
\end{aligned}$$

From Proposition 2.23, we conclude that if ϕ is given as in Theorem 2.5, ϕ is harmonic and has values in $G_*(\mathbb{C}^n)$. It remains to prove the converse, that all harmonic maps into $G_*(\mathbb{C}^n)$ with finite uniton number can be given this way.
If ϕ has uniton number 1, $\phi = (\pi_{F_0} - \pi_{F_0}^\perp)(\pi_1 - \pi_1^\perp)$. Hence, ϕ lies in $G_*(\mathbb{C}^n)$ if and only if F_0 splits $\underline{\alpha}_1$. But $\underline{\alpha}_1$ is spanned by some collection of $H_{i,j}$. Hence, it must be that we can choose the spanning set taking values either in F_0 or in $F_0^\perp$. In that case, $\phi = \pi_{F_1} - \pi_{F_1}^\perp$. If ϕ has uniton number 2, then $\phi = (\pi_{F_1} - \pi_{F_1}^\perp)(\pi_2 - \pi_2^\perp)$. Hence, ϕ takes values in $G_*(\mathbb{C}^n)$ if and only if $\underline{F}_1$ splits $\underline{\alpha}_2$. But $\underline{\alpha}_2$ is spanned by vectors of the form $H_{0,j} + \pi_1^\perp H_{1,j}$ (and $A_z^{\phi_1}(H_{0,j} + \pi_1^\perp H_{1,j})$). Since $\underline{F}_1$ splits $\underline{\alpha}_2$, we must have $\pi_{F_1}(\underline{\alpha}_2)$ and $\pi_{F_1}^\perp(\underline{\alpha}_2)$ lying in $\underline{\alpha}_2$. Now, if $H_{0,j}$ lies in F_0 $(\pi_{F_0}^\perp H_{0,j} = 0)$, then it lies in $F_0 \cap \underline{\alpha}_1 \subseteq \underline{F}_1$. Hence,

$$\pi_{F_1}^\perp(H_{0,j} + \pi_1^\perp H_{1,j}) = \pi_{F_1}^\perp(H_{0,j} + \pi_1^\perp\pi_{F_0}^\perp H_{1,j} + \pi_1^\perp\pi_{F_0} H_{1,j}) = \pi_1^\perp\pi_{F_0} H_{1,j}$$

lies in $\underline{\alpha}_2$. Write $\tilde{H}_1 = \pi_{F_0}(H_{1,j})$. Then, $\underline{\alpha}_2$ is spanned by $\pi_1^\perp \tilde{H}_{i,j}$ and $H_{0,j} + \pi_1^\perp \hat{H}_{1,j}$, where $\hat{H}_{1,j} = H_{1,j} - \tilde{H}_{1,j}$ lies in $F_0^\perp$.
In general, assume that $\pi_{F_0}^\perp H_{0,j} = 0$ and r is odd (the remaining cases are similar). Write

$$\begin{aligned}\alpha_{r+1,j}^{(0)} &= \alpha_{r,j}^{(0)} + \pi_r^\perp\Big(\sum_{t=0}^{r-1} C_t^{r-1} H_{t+1,j}\Big)\\ &= \alpha_{r,j}^{(0)} + \pi_r^\perp\Big(\sum_{t=0}^{r-1} S_t^{r-1}(\pi_{F_0} + \pi_{F_0}^\perp)\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big).\end{aligned}$$

By the induction hypothesis, $\alpha_{r,j}^{(0)}$ lies in $\underline{F}_r$. By Lemma 2.22, if t is even,

$$\pi_r^\perp\Big(S_t^{r-1}\pi_{F_0}\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big)$$

lies in $\underline{F}_r^\perp$ and

$$\pi_r^\perp\Big(S_t^{r-1}\pi_{F_0}^\perp\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big)$$

lies in $\underline{F}_r$. For t odd, changing the roles of F_0 and $F_0^\perp$ we get the same conclusion.
Since $\underline{F}_r$ splits $\underline{\alpha}_{r+1}$, we must have $\pi_{F_r}^\perp(\alpha_{r+1,j}^{(0)})$ and $\pi_{F_r}(\alpha_{r+1,j}^{(0)})$ in $\underline{\alpha}_{r+1}$. But

$$\pi_{F_r}^\perp(\alpha_{r+1,j}^{(0)}) = \pi_r^\perp\Big(\sum_{\substack{t=0\\ t\text{ even}}}^{r-1} S_t^{r-1}\pi_{F_0}\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big) + \pi_r^\perp\Big(\sum_{\substack{t=0\\ t\text{ odd}}}^{r-1} S_t^{r-1}\pi_{F_0}^\perp\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j}\Big)$$

lies in $\underline{\alpha}_{r+1}$. By the induction hypothesis,

$$\sum_{\substack{t=0\\ t\text{ even}}}^{r-1}\pi_{F_0}\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j} = S_{r-1}^{r-1}\pi_{F_0}\sum_{s=0}^{r-1}\binom{r-1}{s} H_{s+1,j}$$

and

$$\sum_{\substack{t=0\\ t\text{ odd}}}^{r-1}\pi_{F_0}^\perp\sum_{s=0}^{s}\binom{t}{s} H_{s+1,j} = \sum_{\substack{t=0\\ t\text{ odd}}}^{r-2}\pi_{F_0}^\perp\sum_{s=0}^{t}\binom{t}{s} H_{s+1,j} = 0.$$

Hence

$$\pi_r^\perp S_{r-1}^{r-1}\pi_{F_0}\sum_{s=0}^{r-1}\binom{r-1}{s} H_{s+1,j} = C_r^r\pi_{F_0}\sum_{s=0}^{r-1}\binom{r-1}{s} H_{s+1,j}$$

lies in $\underline{\alpha}_{r+1}$.
Take $\tilde{H}_j$ the holomorphic vector field given by

$$\tilde{H}_j = \pi_{F_0}\sum_{s=0}^{r-1}\binom{r-1}{s} H_{s+1,j}.$$

Then we can write

$$\begin{aligned}\pi_{F_r}(\alpha^{(0)}_{r+1,j}) &= \alpha^{(0)}_{r+1,j} - \pi^{\perp}_{F_r}(\alpha^{(0)}_{r+1,j})\\ &= \alpha^{(0)}_{r+1,j} - \pi^{\perp}_{r}\left(C^{r-1}_0 H_{1,j} + C^{r-1}_1 H_{2,j} + \ldots + C^{r-1}_{r-1}(H_{r,j} - \tilde{H}_{r,j})\right).\end{aligned}$$

Writing $\hat{H}_{r,j} = H_{r,j} - \tilde{H}_{r,j}$, we have

$$\operatorname{span}\{\alpha^{(0)}_{r+1,j}\} = \operatorname{span}\{C^r_r \tilde{H}_{r,j}, C^r_0 H_{0,j} + \ldots + C^r_{r-1} H_{r-1,j} + C^r_r \hat{H}_{r,j}\}.$$

We shall check that this new holomorphic data satisfies our conditions. As a matter of fact, $\pi^{\perp}_{F_0}(H_{0,j}) = 0$ and

$$\pi_{F_0}\left(\sum_{s=1}^{r-1}\binom{r-1}{s-1}H_{s,j} + \hat{H}_{r,j}\right) = \pi_{F_0}\left(\sum_{s=1}^{r}\binom{r-1}{s-1}H_{s,j}\right) - \tilde{H}_{r,j} = 0.$$

Also, $\pi_{F_0}(0) = 0$ and $\pi^{\perp}_{F_0}(\tilde{H}_{r,j}) = 0$, concluding our proof. □

Proof of Proposition 2.3. We must show that $\phi \cap \underline{\alpha}$ and $\phi \cap \underline{\alpha}^{\perp}$ are, respectively, holomorphic and anti-holomorphic subbundles of $(\mathbb{C}^n, D^{\phi}_{\bar{z}})$. From Proposition 2.23 we know $D^{\phi}_{\bar{z}}$-holomorphic basis for $\phi \cap \underline{\alpha}$ and for $\phi \cap \underline{\alpha}^{\perp}$. Hence,

$$D^{\phi}_{\bar{z}}(\phi \cap \underline{\alpha}) \subseteq \phi \cap \underline{\alpha} \text{ and } D^{\phi}_{\bar{z}}(\phi \cap \underline{\alpha}^{\perp}) \subseteq \phi \cap \underline{\alpha}^{\perp}.$$

Since $D^{\phi}_{z}\underline{\alpha}^{\perp} \subseteq \underline{\alpha}^{\perp}$, the result follows from the identity

$$< D^{\phi}_{z}(\phi^{\perp} \cap \underline{\alpha}^{\perp}), \phi \cap \underline{\alpha}^{\perp} >=< \phi^{\perp} \cap \underline{\alpha}^{\perp}, D^{\phi}_{\bar{z}}(\phi \cap \underline{\alpha}^{\perp}) >= 0.$$

□

In order to prove Theorem 2.11, we start with the following Lemma.

Lemma 2.24. *Let $r \in \{1, \ldots, n-1\}$, F_0 be a k-dimensional subspace of $\mathbb{C}^n$ and consider a pair (L, S) adapted to F_0. For any F_0-array $(K_{i,j})_{0\le i\le r-1, 1\le j\le n}$ which matches (L, S),*

(i) $\operatorname{rank}(\underline{\alpha}_{i+1}) = \operatorname{rank}(\underline{\alpha}_i) + \sum_{t=0}^{i}(l^{i-t}_t + s^{i-t}_t)$;

(ii) $\operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i) = \begin{cases}\operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1}) + \sum_{t=0}^{i} l^{i-t}_t, \text{ if } i \text{ even}\\ \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1}) + \sum_{t=0}^{i} s^{i-t}_t, \text{ if } i \text{ odd};\end{cases}$

(iii) $\operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}^{\perp}_i) = \begin{cases}\operatorname{rank}(\underline{\alpha}_i \cap \underline{F}^{\perp}_{i-1}) + \sum_{t=0}^{i} s^{i-t}_t, \text{ if } i \text{ even}\\ \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}^{\perp}_{i-1}) + \sum_{t=0}^{i} l^{i-t}_t, \text{ if } i \text{ odd}.\end{cases}$

for each $i \in \{1, \ldots, r-1\}$.

Proof. Let $i \in \{1, ..., r-1\}$. We know that $\underline{\alpha}_{i+1}$ is spanned by $\{\underline{\alpha}^{(k)}_{i+1,j}\}_{0\le k\le i, 1\le j\le n}$. Since our array matches the pair (L, S), we split $\underline{\alpha}_{i+1}$, considering $\underline{\alpha}_{i+1} = P \oplus Q$, where

$$P = \operatorname{span}\left\{\underline{\alpha}^{(k)}_{i+1,j_k}\right\}_{\substack{0\le k\le i-1\\ 1\le j_k\le A_{i-k}}} \text{ and } Q = \operatorname{span}\left\{C^i_i H^{(k)}_{i-k,j_k}\right\}_{\substack{0\le k\le i\\ A_{i-k}+1\le j_k\le A_{i-k+1}}}.$$

The matching condition tells us that $K_{l,j} = 0$ whenever $l < i$ and $A_i \le j \le A_{i+1}$. Then, for every $0 \le k \le i$ and $A_{i-k}+1 \le j_k \le A_{i-k+1}$, we have $C^i_i H^{(k)}_{i-k,j_k} = C^i_i K^{(k)}_{i-k,j_k}$; hence, from Definition 2.9, $\operatorname{rank}(Q) = \sum_{j=0}^{i} l^{i-j}_j$, if i even, and $\operatorname{rank}(Q) = \sum_{j=0}^{i} s^{i-j}_j$, if i odd.
The matching condition ensures also that, whenever $0 \le k \le i-1$ and $1 \le j \le A_{i-k}$, $\alpha^{(k)}_{i,j} = \sum_{l=k}^{i-1} C^{i-1}_l H^{(k)}_{l-k,j} \neq 0$. Writing, as before, $\alpha^{(k)}_{i+1,j} = \alpha^{(k)}_{i,j} + \pi^{\perp}_i \left(\sum_{l=0}^{i-1} S^{i-1}_l \sum_{t=0}^{l}\binom{l}{t} H_{t+1,j}\right)$, we conclude that $\operatorname{rank}(P) = \operatorname{rank}(\alpha_{i,j})$, proving (i). On the other hand, since $\underline{F}_i = \underline{\alpha}_i \cap \underline{F}_{i-1} \oplus \underline{\alpha}^{\perp}_i \cap \underline{F}^{\perp}_{i-1}$, we also obtain $\operatorname{rank}(P \cap \underline{F}_i) = \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1})$, thus $\operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i) = \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1}) + \operatorname{rank}(Q)$ getting (ii). The proof of (iii) is analogous, just interchanging $\underline{F}$ with $\underline{F}^{\perp}$ and l with s. □

From the previous Lemma, using an induction argument, we easily get the following identities:

Corollary 2.25. *Under the above conditions the following equalities hold:*

$$\text{(i) } \operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i) = \begin{cases} \displaystyle\sum_{j=0}^{\frac{i}{2}} \sum_{t=0}^{2j} l^{2j-t}_t + \sum_{j=0}^{\frac{i-2}{2}} \sum_{t=0}^{2j+1} s^{2j+1-t}_t, & \text{if } i \text{ even} \\ \displaystyle\sum_{j=0}^{\frac{i-1}{2}} \sum_{t=0}^{2j} l^{2j-t}_t + \sum_{j=0}^{\frac{i-1}{2}} \sum_{t=0}^{2j+1} s^{2j+1-t}_t, & \text{if } i \text{ odd;} \end{cases}$$

$$\text{(ii) } \operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}^{\perp}_i) = \begin{cases} \displaystyle\sum_{j=0}^{\frac{i}{2}} \sum_{t=0}^{2j} s^{2j-t}_t + \sum_{j=0}^{\frac{i-2}{2}} \sum_{t=0}^{2j+1} l^{2j+1-t}_t, & \text{if } i \text{ even} \\ \displaystyle\sum_{j=0}^{\frac{i-1}{2}} \sum_{t=0}^{2j} s^{2j-t}_t + \sum_{j=0}^{\frac{i-1}{2}} \sum_{t=0}^{2j+1} l^{2j+1-t}_t, & \text{if } i \text{ odd.} \end{cases}$$

Proof of Theorem 2.11.
From our data, we have, of course, $\operatorname{rank}(\underline{\alpha}_1 \cap F_0) = l^0_0$ and $\operatorname{rank}(\underline{\alpha}_1 \cap F^{\perp}_0) = s^0_0$. Hence, since $\underline{F}_1 = \underline{\alpha}_1 \cap F_0 \oplus \underline{\alpha}^{\perp}_1 \cap F^{\perp}_0$, $\operatorname{rank}(\underline{F}_1) = l^0_0 + n - k - \operatorname{rank}(\underline{\alpha}_1 \cap F^{\perp}_0) = n - [k + s^0_0 - l^0_0]$.

Analogously, from the equality $\underline{F}_2 = \underline{\alpha}_2 \cap \underline{F}_1 \oplus \underline{\alpha}_2^{\perp} \cap \underline{F}_1^{\perp}$, we get, using Lemma 2.24 and Corollary 2.25

$$\begin{aligned}\operatorname{rank}(\underline{F}_2) &= l_0^0 + s_0^1 + s_1^0 + \operatorname{rank}(\underline{F}_1^{\perp}) - \operatorname{rank}(\underline{\alpha}_2 \cap \underline{F}_1^{\perp}) \\ &= l_0^0 + s_0^1 + s_1^0 + k + s_0^0 - l_0^0 - (s_0^0 + l_0^1 + l_1^0) \\ &= k + \sum_{j=0}^{1}(s_j^{2t+1-j} - l_j^{2t+1-j}).\end{aligned}$$

Assume now that the proposition holds for $\underline{F}_i$, where i is even (the proof for i odd is analogous). The equality $\underline{F}_{i+1} = \underline{\alpha}_{i+1} \cap \underline{F}_i \oplus \underline{\alpha}_{i+1}^{\perp} \cap \underline{F}_i^{\perp}$ implies that

$$\begin{aligned}\operatorname{rank}(\underline{F}_{i+1}) &= \operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i) + \operatorname{rank}(\underline{\alpha}_{i+1}^{\perp} \cap \underline{F}_i^{\perp}) \\ &= \operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i) + \operatorname{rank}(\underline{F}_i^{\perp}) - \operatorname{rank}(\underline{\alpha}_{i+1} \cap \underline{F}_i^{\perp}).\end{aligned}$$

Now, using Lemma 2.24, we get

$$\operatorname{rank}(\underline{F}_{i+1}) = \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1}) + \sum_{t=0}^{i} l_t^{i-t} + n - \operatorname{rank}(\underline{F}_i) - \operatorname{rank}(\underline{\alpha}_i \cap \underline{F}_{i-1}^{\perp}) - \sum_{t=0}^{i} s_t^{i-t}.$$

From Corollary 2.25 and the knowledge of $\operatorname{rank}(\underline{F}_i)$, we conclude that

$$\begin{aligned}\operatorname{rank}(\underline{F}_{i+1}) &= n - k - \sum_{j=0}^{\frac{i}{2}-1}\sum_{t=0}^{2j+1}(s_t^{2j+1-t} - l_t^{2j+1-t}) + \sum_{j=0}^{\frac{i-2}{2}}\sum_{t=0}^{2j}(l_t^{2j-t} - s_t^{2j-t}) \\ &+ \sum_{j=0}^{\frac{i-2}{2}}\sum_{t=0}^{2j+1}(s_t^{2j+1-t} - l_t^{2j+1-t}) + \sum_{j=0}^{i}(l_j^{i-j} - s_j^{i-j}) \\ &= n - \left[k + \sum_{j=0}^{\frac{i}{2}-1}\sum_{t=0}^{2j}(s_t^{2j-t} - l_t^{2j-t})\right] + \sum_{j=0}^{i}(l_j^{i-j} - s_j^{i-j}) \\ &= n - \left[k + \sum_{j=0}^{\frac{i}{2}}\sum_{t=0}^{2j}(s_j^{2t-j} - l_j^{2t-j})\right],\end{aligned}$$

as wanted. □

Proof of Theorem 2.14.
Given a matrix D, we will let D_i denote its i'th column. We consider $k \in \{1, ..., n\}$ fixed and let r_k denote the maximal uniton number for harmonic maps $\varphi = (\pi_0 - \pi_0^{\perp})(\pi_1 - \pi_1^{\perp})...(\pi_i - \pi_i^{\perp}) : S^2 \to G_*(\mathbb{C}^n)$, where F_0 is a complex subspace of $\mathbb{C}^n$ with dimension k.
We will first show that it is not possible to have simultaneously $r_k \geq k$ and $r_k \geq n-k$. Indeed, these two conditions would imply the existence of a pair (L, S) of $r_k \times r_k$ matrices, adapted to

F_0, matching a given array; from the fullness of $\underline{\alpha}_1$ we would get

$$L_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots \\ 1\end{matrix}}\right\} n-k \\ \vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}} \end{matrix} \quad \text{and } S_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots \\ 1\end{matrix}}\right\} k \\ \vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}} \end{matrix}$$

which cannot happen since the sum of the entries of both matrices must be strictly less than n.
We have to analyze, separately, the different situations $k < p$ and $k \geq p$. The techniques are similar, so that we only present the first case.
Consider that $k < p$ and let (L, S) be a pair of $r_k \times r_k$ matrices, adapted to F_0 and matching an F_0-array. It is easily seen that $k < 2p - k \leq n - k$. Consider $k < r_k \leq n - k$. Assume that r_k is even. From Theorem 2.11, we can write

$$p - k = \sum_{j=0}^{\frac{r_k}{2}-1} \sum_{t=0}^{2j+1} (s_t^{2j+1-t} - l_t^{2j+1-t}).$$

The fullness of $\underline{\alpha}_1$ implies

$$L_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \vdots \\ 1\end{matrix}}\right\} k \\ \vphantom{\begin{matrix}0\\ \vdots \\ 0\end{matrix}} \end{matrix}, \; L_i = 0, \text{ if } i > 1 \text{ and } S_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

so that

$$\sum_{j=0}^{\frac{r_k}{2}-1} (s_0^{2j+1} - l_0^{2j+1}) = \frac{r_k - k + a_k}{2}.$$

Hence, $p - k = \frac{r_k - k + a_k}{2} + \theta$, where $0 \leq \theta \leq n - r_k - 1$. Therefore $p - k \geq \frac{r_k - k + a_k}{2}$, which implies $r_k \leq 2p - k - a_k$.
If r_k is odd we will get instead

$$\begin{aligned} n - (p + k) &= \sum_{j=0}^{\frac{r_k - 1}{2}} \sum_{t=0}^{2j} (s_t^{2j-t} - l_t^{2j-t}) \\ &= \frac{r_k - k + 1 - a_k}{2} + \sum_{j=0}^{\frac{r_k - 1}{2}} \sum_{t=1}^{2j} (s_t^{2j} - l_t^{2j}) < \frac{r_k - k + 1 - a_k}{2} + n - k - r_k. \end{aligned}$$

Hence $r_k < 2p - k - a_k + 1$, or $r_k \leq 2p - k - a_k$.

These estimates are sharp as we may easily see. For instance, in the case r_k odd, we can consider the pair (L,S) of order $2p-k-a_k$ with

$$L_1=\begin{bmatrix}1\\ \vdots\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix}\begin{matrix}\left.\begin{matrix}\\ \\ \\ \end{matrix}\right\}k\\ \\ \\ \\ \end{matrix},\ L_i=S_i=0 \text{ if } i>1 \text{ and } S_1=\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}.$$

Taking meromorphic functions $L_{0,1}$ and $E_{0,1}$ with such that

$$\mathrm{span}\{L_{0,1},L_{0,1}^{(1)},...,L_{0,1}^{(k)}\}=F_0 \text{ and } \mathrm{span}\{E_{0,1},E_{0,1}^{(1)},...,E_{0,1}^{(n-k)}\}=F_0^{\perp},$$

we get an array matching (L,S).

Then $\sum_{j=0}^{\frac{r_k-1}{2}}\sum_{t=0}^{2j}(s_t^{2j-t}-l_t^{2j-t})=\frac{2p-k-a_k-k+a_k}{2}=p-k$, concluding the proof.

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